Sine-Gordon description of the scaling three-state Potts antiferromagnet on the square lattice

2001 J. Phys. A: Math. Gen. 34 L311

(http://iopscience.iop.org/0305-4470/34/21/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L311–L317 www.iop.org/Journals/ja PII: S0305-4470(01)23800-1

LETTER TO THE EDITOR

Sine–Gordon description of the scaling three-state Potts antiferromagnet on the square lattice

Gesualdo Delfino

International School for Advanced Studies and Istituto Nazionale di Fisica Nucleare, via Beirut 4, 34014 Trieste, Italy

E-mail: delfino@sissa.it

Received 5 April 2001

Abstract

The scaling limit as $T \to 0$ of the antiferromagnetic three-state Potts model on the square lattice is described by the sine–Gordon quantum field theory at a specific value of the coupling. We show that the correspondence follows unambigously from an analysis of the sine–Gordon operator space based on locality, and that the scalar operators carrying solitonic charge play an essential role in the description of the lattice model. We then evaluate the correlation functions within the form factor approach and give a number of universal predictions that can be checked in numerical simulations.

PACS numbers: 0550, 7550E, 7510, 6460, 1110

The study of antiferromagnetic models is a notoriously difficult problem of statistical mechanics. The dependence on the lattice structure produces a variety of behaviours much richer than in the ferromagnetic case and forces a case-by-case analysis. Many antiferromagnetic models possess a critical point (often at zero temperature), so that their continuum limit can also be investigated through field theoretical methods. Among them, the square-lattice three-state Potts model has been the object of both numerical and theoretical studies. While this model has been known for a long time to be critical at zero temperature, the issue of the approach to criticality presents several subtleties, including the identification of the correct scaling variable. Very recently, the authors of [1] exploited a mapping onto a height model to identify the excitations on the lattice at nonzero temperature and explain the anomalous corrections to scaling previously observed in Monte Carlo studies [2].

It is the purpose of this Letter to point out that the *scaling limit* of the square-lattice three-state Potts antiferromagnet is *exactly* solvable due to its equivalence to a specific point of the sine–Gordon model (an integrable quantum field theory), and to derive from this continuum approach some universal predictions that can be checked through simulations on the lattice.

As often happens for two-dimensional systems exhibiting a Gaussian critical point, a relation of the scaling limit to the sine–Gordon model is expected. The actual task is that of understanding what the symmetries and the operators of the lattice model become in the field theoretic language. Here we will show that these identifications follow in a quite natural way from the analysis of the sine–Gordon operator space based on locality, and that the topologically charged scalar operators play a major role in the continuum description of the lattice model. Matrix elements for these operators were first computed in [3], where they arise in the description of the scaling Ashkin–Teller model. We will then illustrate how the correspondence at the operatorial level translates in the language of particle excitations, in order to exploit the exact S-matrix solution of the sine–Gordon model and extract from that the correlation functions for the operators of physical interest. In particular, we will give a number of specific and universal predictions suitable for a numerical verification on the lattice.

The three-state Potts model is defined by the lattice Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \delta_{s_i,s_j} \tag{1}$$

where the spin variable s_j on the jth lattice site takes the values (colours) 0, 1 and 2 and the sum is over nearest neighbours. The Hamiltonian is invariant under permutations of the colours. Here we are interested in the antiferromagnetic case ($J < 0$) on the square lattice.

At zero temperature, the system is in one of the ground states in which nearest-neighbour spins have different values. This problem is equivalent to the three-colouring problem of the square lattice, which admits an exact mapping onto a specific point of the six-vertex model [4, 5]. From the exact solution of the latter we know that our zero-temperature system is critical and that its long-distance behaviour is described by a free massless boson. Relevant operators that can be identified through the zero-temperature analysis on the lattice [6–9] are the staggered magnetization $\Sigma_j = (-1)^{j_1+j_2}\, \mathrm{e}^{2\mathrm{i}\pi s_j/3}$ with scaling dimension $1/6$, the uniform magnetization $\sigma_j = \mathrm{e}^{2\mathrm{i}\pi s_j/3}$ with scaling dimension $2/3$, and the staggered polarization[1] $\mathcal{P}_j = (-1)^{j_1+j_2} \sum_i' (2\delta_{s_i,s_j} - 1)$ with scaling dimension $3/2$. We identify the jth site of the square lattice through a pair of integers (j_1, j_2), and call the collection of the sites with $j_1 + j_2$ even (odd) the even (odd) sublattice.

At non-zero temperature, the model develops a finite correlation length and its scaling limit is described by the perturbation of the Gaussian fixed point through the continuum version of the thermal operator $\mathcal{E}_j = \sum_i \delta_{s_i,s_j}$. This perturbed theory is the sine–Gordon model defined by the Euclidean action[2]

$$\mathcal{A} = \int \mathrm{d}^2x \, \left(\tfrac{1}{2}\, \partial_\alpha \varphi \partial^\alpha \varphi - \mu \cos \beta\varphi \right) \tag{2}$$

for some value of the coupling β to be determined. The theory (2) describes the scaling limit of several lattice models whose critical point corresponds to a conformal field theory with central charge $C = 1$ (see e.g. [3, 10] for other examples discussed in the framework of this Letter). In order to proceed with the description of the antiferromagnetic Potts model we need to recall a few points about the operator content of the sine–Gordon model.

At criticality ($\mu = 0$), the boson field can be decomposed into its holomorphic and antiholomorphic parts as $\varphi(x) = \phi(z) + \bar{\phi}(\bar{z})$, where we have introduced the complex coordinates $z = x_1 + \mathrm{i}x_2$ and $\bar{z} = x_1 - \mathrm{i}x_2$. The scaling operators of the theory are the vertex operators

$$V_{p,\bar{p}}(x) = \mathrm{e}^{\mathrm{i}[p\phi(z) + \bar{p}\bar{\phi}(\bar{z})]} \tag{3}$$

[1] The primed sum indicates summation over the next nearest neighbours of j.
[2] Throughout this Letter we use the standard notation β for the sine–Gordon dimensionless coupling. No confusion with the inverse temperature of the lattice model should be made.

with conformal dimensions $(\Delta, \bar{\Delta}) = (p^2/8\pi, \bar{p}^2/8\pi)$ and spin $s = \Delta - \bar{\Delta}$. They satisfy the Gaussian operator product expansion

$$V_{p_1,\bar{p}_1}(x)V_{p_2,\bar{p}_2}(0) = z^{p_1p_2/4\pi}\bar{z}^{\bar{p}_1\bar{p}_2/4\pi}\, V_{p_1+p_2,\bar{p}_1+\bar{p}_2}(0) + \cdots. \tag{4}$$

We see from this relation that taking $V_{p_1,\bar{p}_1}(x)$ around $V_{p_2,\bar{p}_2}(0)$ by sending $z \to z\mathrm{e}^{2\mathrm{i}\pi}$ and $\bar{z} \to \bar{z}\mathrm{e}^{-2\mathrm{i}\pi}$ produces a phase factor $\mathrm{e}^{2\mathrm{i}\pi\gamma_{1,2}}$, where

$$\gamma_{1,2} = \frac{1}{4\pi}\,(p_1p_2 - \bar{p}_1\bar{p}_2) \tag{5}$$

is called the index of mutual locality. If $\gamma_{1,2}$ is an integer the correlator $\langle V_{p_1,\bar{p}_1}(x)V_{p_2,\bar{p}_2}(0)\rangle$ is single valued and the two operators are said to be mutually local. Since $\gamma_{1,1} = 2s$, the operators which are local with respect to themselves (the only ones we are interested in here) must have integer or half-integer spin.

In the off-critical theory (2), the operators which are local with respect to the perturbing operator $\cos\beta\varphi \sim V_{\beta,\beta} + V_{-\beta,-\beta}$ form a 'local sector' into which all the operators of interest for the description of the lattice model are expected to fall. This locality requirement selects the operators $V_{p,\bar{p}}$ with $p - \bar{p} = 4\pi m/\beta$, m integer, namely

$$V_p(x) \equiv V_{p,p}(x) = \mathrm{e}^{\mathrm{i}p\varphi(x)} \tag{6}$$

and

$$U_{n,m}(x) \equiv V_{\frac{n\beta}{2m}+\frac{2\pi}{\beta}m,\,\frac{n\beta}{2m}-\frac{2\pi}{\beta}m}(x) = \mathrm{e}^{\mathrm{i}[\frac{n\beta}{2m}\varphi(x)+\frac{2\pi}{\beta}m\tilde{\varphi}(x)]}$$
$$n = 2s = 0, \pm1, \ldots \qquad m = \pm1, \ldots. \tag{7}$$

Here we have introduced the 'dual' boson field $\tilde{\varphi}$, which is $\phi(z) - \bar{\phi}(\bar{z})$ at criticality and satisfies the relation

$$\mathrm{i}\frac{\partial\tilde{\varphi}}{\partial x_\alpha} = \varepsilon_{\alpha\beta}\frac{\partial\varphi}{\partial x_\beta}. \tag{8}$$

The operators V_p and $U_{0,m}$ are scalars ($s = 0$) and have scaling dimensions $X_p = p^2/4\pi$ and $X_{0,m} = \pi m^2/\beta^2$, respectively ($X = \Delta + \bar{\Delta}$). The action (2) describes a relevant perturbation of the Gaussian fixed point for $\beta^2 < 8\pi$; in this range the only operators $U_{0,m}$ which are relevant ($X_{0,m} < 2$) are those with $|m| \leqslant 3$.

The lowest operators with $|s| = 1/2$, i.e. $\Psi = U_{\pm1,1}$ and $\Psi^* = U_{\pm1,-1}$, are complex conjugate two-component spinors with conformal dimensions $\Delta_{n,m}$ given by

$$\Delta_{\pm1,1} = \Delta_{\pm1,-1} = \frac{1}{8}\left(\frac{\beta^2}{4\pi} \pm 2 + \frac{4\pi}{\beta^2}\right) \tag{9}$$

$$\bar{\Delta}_{\pm1,1} = \bar{\Delta}_{\pm1,-1} = \frac{1}{8}\left(\frac{\beta^2}{4\pi} \mp 2 + \frac{4\pi}{\beta^2}\right). \tag{10}$$

It has been known since Coleman [11] that the sine–Gordon model is equivalent to the theory of a Dirac fermion with four-fermion interaction, the Thirring model. The expression for Ψ concides with the bosonization formula for the Thirring fermion originally derived by Mandelstam [12]. The value $\beta^2 = 4\pi$ for which the dimensions (9) and (10) take the free fermionic values corresponds to the free point of the Thirring model.

It follows from these considerations that the integer m in (7) is a fermionic charge. Since the Thirring fermions correspond to the solitons interpolating between adjacent vacua of the periodic bosonic potential, we will also call m the topologic charge. Hence, we can make a distinction between neutral scalar operators $V_p(x)$ with a non-zero vacuum expectation value, and charged scalar operators $U_{0,m}(x)$.

Table 1. Relevant operators on the lattice and their continuum counterparts in the sine-Gordon model.

Φ	Lattice definition	Continuum limit	X_Φ	Topologic charge
$\mathcal{E}$	$\sum_i \delta_{s_i,s_j}$	$\cos\sqrt{6\pi}\varphi$	3/2	0
Σ	$(-1)^{j_1+j_2}\,e^{2i\pi s_j/3}$	$e^{i\sqrt{2\pi/3}\tilde{\varphi}}$	1/6	1
σ	$e^{2i\pi s_j/3}$	$e^{-2i\sqrt{2\pi/3}\tilde{\varphi}}$	2/3	−2
$\mathcal{P}$	$(-1)^{j_1+j_2}\sum_i'(2\delta_{s_i,s_j}-1)$	$\cos\sqrt{6\pi}\,\tilde{\varphi}$	3/2	±3

Going back to the Potts model it is not difficult to identify the continuum limit of the lattice operators among the sine–Gordon operators. Both the staggered and uniform magnetization are charged with respect to colour permutations and must be found among the $U_{0,m}$. Comparison with the known scaling dimensions gives $\Sigma(x) \sim U_{0,1}(x)$ and $\sigma(x) \sim U_{0,-2}(x)$, and selects

$$\beta = \sqrt{6\pi} \tag{11}$$

as the value for which the sine–Gordon model (2) describes the scaling limit of the three-state Potts antiferromagnet on the square lattice. This immediately fixes the scaling dimension of the thermal operator $\mathcal{E}(x) \sim \cos\beta\varphi(x)$ to be 3/2, in agreement with the result obtained in [1] by studying the vortex excitations on the lattice.

On the basis of these identifications we can see how the symmetries of the lattice model translate in the sine–Gordon language. The group S_3 of colour permutations can be decomposed into the Z_3 transformations associated with cyclic permutations plus the complex conjugation of $e^{2i\pi s_j/3}$. The latter operation simply corresponds to the complex conjugation of the sine–Gordon exponentials, while the elementary Z_3 transformation maps into the shift $\tilde{\varphi} \to \tilde{\varphi} + 2\pi/\beta$. The Z_3 charge coincides with the topologic charge m (mod 3).

The lattice operators are also characterized by their parity under the transformation which exchanges the even and odd sublattices: $\mathcal{E}_j$ and σ_j are even, while Σ_j and $\mathcal{P}_j$ are odd. It appears that in the continuum limit this parity property corresponds to $(-1)^m$. The staggered polarization $\mathcal{P}_j$ is invariant under colour permutations and must correspond to the operator $U_{0,3} + U_{0,-3}$, which has indeed the expected scaling dimension 3/2. We summarize the situation in table 1.

The equivalence with a particular case of the sine–Gordon model ensures that the scaling limit of the lattice model is an *integrable* quantum field theory whose associated scattering theory is known exactly [13]. The elementary excitations are a pair of conjugated particles A_+ and A_- (the sine–Gordon soliton and antisoliton). The fact that $\beta^2 = 6\pi$ falls in the repulsive sine–Gordon regime $\beta^2 > 4\pi$ ensures that no other particles are present in the spectrum. It can be interesting to mention that the Z_3-preserving fusion $A_+A_+ \to A_-$ is forbidden here because it violates the sublattice parity $(-1)^m$; it is instead characteristic of the scattering theory of the ferromagnetic case [14, 15] in which the lattice plays no role.

Due to the factorization of multiparticle scattering in the integrable quantum field theories, the scattering theory is completely determined by the two-particle S-matrix defined by the relation[3]

$$A_a(\theta_1)A_b(\theta_2) = \sum_{c,d=\pm} S_{ab}^{cd}(\theta_1-\theta_2)A_d(\theta_2)A_c(\theta_1) \qquad a,b=\pm. \tag{12}$$

[3] The rapidity θ parametrizes the on-shell energy and momentum of a particle as $(p^0, p^1) = (m\cosh\theta, m\sinh\theta)$.

The non-zero scattering amplitudes are given by[4] [13]

$$S_{++}^{++}(\theta) = S_{--}^{--}(\theta) = S_0(\theta) \tag{13}$$

$$S_{+-}^{+-}(\theta) = S_{-+}^{-+}(\theta) = -\frac{\sinh\frac{\pi\theta}{\xi}}{\sinh\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)}\, S_0(\theta) \tag{14}$$

$$S_{+-}^{-+}(\theta) = S_{-+}^{+-}(\theta) = -\frac{\sinh\frac{\mathrm{i}\pi^2}{\xi}}{\sinh\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)}\, S_0(\theta) \tag{15}$$

with

$$S_0(\theta) = -\exp\left\{-\mathrm{i}\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\sinh\frac{x}{2}\left(1-\frac{\xi}{\pi}\right)}{\sinh\frac{x\xi}{2\pi}\cosh\frac{x}{2}}\,\sin\frac{\theta x}{\pi}\right\} \tag{16}$$

$$\xi = \frac{\pi\beta^2}{8\pi - \beta^2}. \tag{17}$$

From the S-matrix one can compute the form factors[5]

$$F_{a_1,\ldots,a_n}^{\Phi}(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|A_{a_1}(\theta_1),\ldots,A_{a_n}(\theta_n)\rangle \qquad a_i = \pm \tag{18}$$

which in turn determine the spectral decomposition of the correlation functions. The form factors satisfy the equations [16–18]

$$F_{a_1,\ldots,a_i,a_{i+1},\ldots,a_n}^{\Phi}(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = \sum_{b_i,b_{i+1}=\pm} S_{a_i,a_{i+1}}^{b_i,b_{i+1}}(\theta_i - \theta_{i+1})F_{a_1,\ldots,b_{i+1},b_i,\ldots,a_n}^{\Phi}(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n) \tag{19}$$

$$F_{a_1,\ldots,a_n}^{\Phi}(\theta_1 + 2\mathrm{i}\pi,\theta_2,\ldots,\theta_n) = \mathrm{e}^{2\mathrm{i}\pi\gamma_{\Phi,a_1}}F_{a_2,\ldots,a_n,a_1}^{\Phi}(\theta_2,\ldots,\theta_n,\theta_1) \tag{20}$$

where $\gamma_{\Phi,a}$ in the last equation is the index of mutual locality between the scalar operator $\Phi(x)$ and the soliton (for $a = +$) or the antisoliton (for $a = -$). Since these particles are created by the operators $U_{0,\pm 1}(x)$, equation (5) gives for the scalar operators

$$\gamma_{V_p,\pm} = \gamma_{V_p;U_{0,\pm 1}} = \pm\frac{p}{\beta} \tag{21}$$

$$\gamma_{U_{0,m};\pm} = \gamma_{U_{0,m};U_{0,\pm 1}} = 0. \tag{22}$$

Equations (19) and (20) can be used to determine the 'lowest' form factors for the operator Φ, i.e. the non-vanishing matrix elements (18) with the smallest n ($n > 0$) which fix the initial conditions of the form factor bootstrap[6]. The lowest form factors for the operators of interest in this Letter are

$$F_{\pm\mp}^{\cos\beta\varphi}(\theta_1,\theta_2) = c_0\,\frac{\cosh\frac{\theta_{12}}{2}}{\sinh\frac{\pi}{2\xi}(\theta_{12} - \mathrm{i}\pi)}\,F_0(\theta_{12}) \tag{23}$$

$$F_{-\mathrm{sg}(m),\ldots,-\mathrm{sg}(m)}^{U_{0,m}}(\theta_1,\ldots,\theta_{|m|}) = c_{|m|}\prod_{i<j}F_0(\theta_{ij}) \tag{24}$$

where $\theta_{ij} \equiv \theta_i - \theta_j$, $\mathrm{sg}(m)$ denotes the sign of m and the c_m are normalization constants. The function

$$F_0(\theta) = -\mathrm{i}\sinh\frac{\theta}{2}\,\exp\left\{\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\sinh\left[\frac{x}{2}\left(1-\frac{\xi}{\pi}\right)\right]}{\sinh\frac{x\xi}{2\pi}\cosh\frac{x}{2}}\,\frac{\sin^2\frac{(\mathrm{i}\pi-\theta)x}{2\pi}}{\sinh x}\right\} \tag{25}$$

[4] For reasons that will become clear later, it is useful to give the results referring to generic values of β in the sine–Gordon model, it being understood that the scaling limit we are dealing with corresponds to $\beta = \sqrt{6\pi}$.

[5] $|0\rangle$ is the vacuum state.

[6] The higher form factors contain additional pairs $A_+(\theta)A_-(\theta')$ in the asymptotic state and are related to the lowest ones by a recursive equation associated with particle–antiparticle annihilation [17].

satisfies the equations

$$F_0(\theta) = S_0(\theta) F_0(-\theta) \tag{26}$$

$$F_0(\theta + 2\mathrm{i}\pi) = F_0(-\theta). \tag{27}$$

The lowest form factors determine the first term in the large-distance expansion of the correlation functions. Using the subscript c to denote the connected correlators we have

$$\langle V_p(x) V_p(0)\rangle_{\mathrm{c}} = \int \frac{\mathrm{d}\theta_1}{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\, |F^{V_p}_{+,-}(\theta_1,\theta_2)|^2\, \mathrm{e}^{-M|x|(\cosh\theta_1+\cosh\theta_2)} + \mathrm{O}(\mathrm{e}^{-4M|x|}) \tag{28}$$

$$\langle U_{0,m}(x) U_{0,-m}(0)\rangle = \frac{1}{m!}\int \frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_m}{2\pi}\, |F^{U_{0,m}}_{-,\ldots,-}(\theta_1,\ldots,\theta_m)|^2\, \mathrm{e}^{-M|x|\sum_{i=1}^{m}\cosh\theta_i} + \mathrm{O}(\mathrm{e}^{-(m+2)M|x|}) \tag{29}$$

where M is the mass of the soliton and we have taken $m > 0$.

There are a number of simple and distinguished universal predictions of the sine–Gordon description that can be tested in numerical simulations of the square-lattice three-state Potts antiferromagnet. The 'exponential' and 'second-moment' correlation lengths ξ_Φ and $\xi_\Phi^{2\mathrm{nd}}$ associated with an operator $\Phi(x)$ are defined as

$$\langle \Phi(x)\Phi^*(0)\rangle_{\mathrm{c}} \sim \exp(-|x|/\xi_\Phi) \qquad |x| \to \infty \tag{30}$$

$$\xi_\Phi^{2\mathrm{nd}} = \left(\frac{1}{4}\,\frac{\int \mathrm{d}^2x\, |x|^2\, \langle\Phi(x)\Phi^*(0)\rangle_{\mathrm{c}}}{\int \mathrm{d}^2x\, \langle\Phi(x)\Phi^*(0)\rangle_{\mathrm{c}}}\right)^{1/2}. \tag{31}$$

The leading behaviour of these quantities near the critical point is

$$\xi_\Phi \simeq f_\Phi\, t^{-\nu} \tag{32}$$

$$\xi_\Phi^{2\mathrm{nd}} \simeq f_\Phi^{2\mathrm{nd}}\, t^{-\nu} \tag{33}$$

where $\nu = 1/(2 - X_{\mathcal{E}}) = 2$, and t measures the deviation from criticality. It appears from numerical simulations [2] and has been discussed in [1] that for this zero-temperature critical point the scaling variable is $t = \mathrm{e}^{J/kT}$. Our operator identifications and equations (28), (29) then give $\xi_\Sigma = 1/M$ and

$$f_\sigma/f_\Sigma = \tfrac{1}{2} \tag{34}$$

$$f_{\mathcal{E}}/f_\Sigma = \tfrac{1}{2} \tag{35}$$

$$f_{\mathcal{P}}/f_\Sigma = \tfrac{1}{3} \tag{36}$$

$$f_\Sigma^{2\mathrm{nd}}/f_\Sigma \approx 1 \tag{37}$$

$$f_\sigma^{2\mathrm{nd}}/f_\sigma \approx 0.439. \tag{38}$$

The last two ratios are evaluated in the lowest form factor approximation, which is known to give extremely accurate results for integrated correlators[7]. We estimate that our error on these two quantities does not exceed 1%.

The universal ratios (34)–(38) have not yet been measured in simulations. Lattice estimates of these quantities would provide the direct confirmation that the universality class of this antiferromagnet is described by the sine–Gordon field theory with $\beta = \sqrt{6\pi}$ and the operator identifications of table 1. In this respect, it is important to mention that the study of the scaling limit of the model on the lattice is complicated in practice by strong corrections to scaling [1, 2]. Cardy *et al* argued in [1] that such corrections are originated by a strictly marginal operator, which couples to the temperature giving an effective thermal dependence

[7] See e.g. [3, 10] for similar computations in other sine–Gordon-related statistical models.

to the stiffness K of the bosonic field, which is related to the sine–Gordon coupling as $\beta = 6\sqrt{K}$. Their estimate of this (nonuniversal) dependence for $K_{\rm eff}(t)$ corresponds to $\beta^2_{\rm eff}(t) = 6\pi - 20.9(4)t - 37.8(2)t^2 + \cdots$. Hence, according to this analysis, the leading corrections to scaling can be fitted by letting the critical exponents and the critical amplitudes vary with $\beta_{\rm eff}$. Concerning the lattice verification of our predictions, the ratios (34)–(37) are not affected by this effect since their values in the sine–Gordon model are determined by symmetry properties of the operators which do not depend on the coupling[8]. The ratio $(f_\sigma^{\rm 2nd}/f_\sigma)_{\rm eff}$ is instead an increasing function of $\beta_{\rm eff}$, so the asymptotic value (38) should be approached from below as $t \to 0$. For example, we find 0.38 when computing this ratio at $\beta^2_{\rm eff}(t = 0.1) \sim 6\pi - 2.47$.

I thank John Cardy for helpful discussions and remarks.

Note added. When this work had been completed [19] appeared, which is devoted to the form factors of the sine–Gordon topologically charged operators and whose main purpose is the determination of their 'conformal' normalization.

References

[1] Cardy J L, Jacobsen J L and Sokal A D 2001 Unusual corrections to scaling in the 3-state Potts antiferromagnet on a square lattice *Preprint* cond-mat/0101197
[2] Ferreira S J and Sokal A D 1999 *J. Stat. Phys.* **96** 461
[3] Delfino G 1999 *Phys. Lett.* B **450** 196
[4] Lieb E H and Wu F Y 1972 *Phase Transitions and Critical Phenomena* vol 1, ed C Domb and M S Green (New York: Academic)
[5] Baxter R J 1982 *Exactly Solved Models of Statistical Mechanics* (London: Academic)
[6] den Nijs M, Nightingale M P and Schick M 1982 *Phys. Rev.* B **26** 2490
[7] Park H and Widom M 1989 *Phys. Rev. Lett.* **63** 1193
[8] Burton J K Jr and Henley C L 1997 *J. Phys. A: Math. Gen.* **30** 8385
[9] Salas J and Sokal A D 1998 *J. Stat. Phys.* **92** 729
[10] Delfino G and Cardy J L 2000 *Phys. Lett.* B **483** 303
[11] Coleman S 1975 *Phys. Rev.* D **11** 2088
[12] Mandelstam S 1975 *Phys. Rev.* D **11** 3026
[13] Zamolodchikov A B and Zamolodchikov Al B 1979 *Ann. Phys., NY* **120** 253
[14] Koberle R and Swieca J A 1979 *Phys. Lett.* B **86** 209
[15] Zamolodchikov A B 1988 *Int. J. Mod. Phys.* A **3** 743
[16] Karowski M and Weisz P 1978 *Nucl. Phys.* B **139** 455
[17] Smirnov F A 1992 *Form Factors in Completely Integrable Models of Quantum Field Theories* (Singapore: World Scientific)
[18] Yurov V P and Zamolodchikov Al B 1991 *Int. J. Mod. Phys.* A **6** 3419
[19] Lukyanov S and Zamolodchikov A 2001 Form factors of soliton creating operators in the sine–Gordon model *Preprint* hep-th/0102079

[8] At least in the range $\beta^2 \geqslant 4\pi$ in which there are no bound states.